

OPTIMAL HARDY–LITTLEWOOD TYPE INEQUALITIES FOR m -LINEAR FORMS ON ℓ_p SPACES WITH $1 \leq p \leq m$

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ABSTRACT. The Hardy–Littlewood inequalities for m -linear forms on ℓ_p spaces are stated for $p > m$. In this paper, among other results, we investigate similar results for $1 \leq p \leq m$. Let \mathbb{K} be \mathbb{R} or \mathbb{C} and $m \geq 2$ be a positive integer. Our main results are the following sharp inequalities:

- (i) If $(r, p) \in ([1, 2] \times [2, 2m]) \cup ([1, \infty) \times [2m, \infty))$, then there is a positive constant $D_{m,r,p}^{\mathbb{K}}$ (not depending on n) such that

$$\left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^r \right)^{\frac{1}{r}} \leq D_{m,r,p}^{\mathbb{K}} n^{\max\{\frac{2mr+2mp-mpr-pr}{2pr}, 0\}} \|T\|$$

for all m -linear forms $T : \ell_p^n \times \dots \times \ell_p^n \rightarrow \mathbb{K}$ and all positive integers n .

- (ii) If $(r, p) \in [2, \infty) \times (m, 2m]$, then

$$\left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^r \right)^{\frac{1}{r}} \leq (\sqrt{2})^{m-1} n^{\max\{\frac{p+mr-rp}{pr}, 0\}} \|T\|$$

for all m -linear forms $T : \ell_p^n \times \dots \times \ell_p^n \rightarrow \mathbb{K}$ and all positive integers n .

Moreover the exponents $\max\{(2mr+2mp-mpr-pr)/2pr, 0\}$ in (i) and $\max\{(p+mr-rp)/pr, 0\}$ in (ii) are optimal. The cases $(r, p) = (2m/(m+1), \infty)$ and $(r, p) = (2mp/(mp+p-2m), p)$ for $p \geq 2m$ and $(r, p) = (p/(p-m), p)$ for $m < p < 2m$ recover the classical Bohnenblust–Hille and Hardy–Littlewood inequalities.

1. INTRODUCTION

The recent years witnessed an intense interest in the Bohnenblust–Hille inequality and its applications in Complex Analysis, Analytic Number Theory and Quantum Information Theory. The Bohnenblust–Hille inequality was proved in 1931, in the *Annals of Mathematics*, as a crucial tool to prove the Bohr’s absolute convergence problem on Dirichlet series. Surprisingly, this inequality was overlooked for almost 80 years and rediscovered some years ago. Since then, it has been used in different areas of Mathematics and several challenging problems remain open. From now on \mathbb{K} denotes the real scalar field \mathbb{R} or the complex scalar field \mathbb{C} .

Theorem (Bohnenblust and Hille [9], 1931). *For any positive integer $m \geq 2$, there exists a constant $B_{\mathbb{K},m} \geq 1$ such that*

$$(1.1) \quad \left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq B_{\mathbb{K},m} \|T\|$$

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for all m -linear forms $T : \ell_\infty^n \times \cdots \times \ell_\infty^n \rightarrow \mathbb{K}$, and all positive integers n . Moreover, the exponent $2m/(m+1)$ is optimal.

For references we mention, for instance, [2, 6, 13, 14, 21] and the very interesting survey [15]. The optimal values of $B_{\mathbb{K},m}$ are unknown; the best known upper and lower estimates for the constants in (1.1) are (see [6] and [18]):

$$\begin{aligned} B_{\mathbb{C},m} &\leq \prod_{j=2}^m \Gamma\left(2 - \frac{1}{j}\right)^{\frac{j}{2-2j}} < m^{\frac{1-\gamma}{2}} < m^{0.22}, \\ 2^{1-\frac{1}{m}} &\leq B_{\mathbb{R},m} \leq 2^{\frac{446381}{55440} - \frac{m}{2}} \prod_{j=14}^m \left(\frac{\Gamma(\frac{3}{2} - \frac{1}{j})}{\sqrt{\pi}}\right)^{\frac{j}{2-2j}} < 1.3 \cdot m^{\frac{2-\log 2-\gamma}{2}} \quad \text{if } m \geq 14, \\ 2^{1-\frac{1}{m}} &\leq B_{\mathbb{R},m} \leq \prod_{j=2}^m 2^{\frac{1}{2j-2}} < 1.3 \cdot m^{\frac{2-\log 2-\gamma}{2}} < 1.3 \cdot m^{0.37} \quad \text{if } m \leq 13, \end{aligned}$$

where γ denotes the Euler–Mascheroni constant.

A natural question is: what happens if we replace ℓ_∞^n by ℓ_p^n in the Bohnenblust–Hille inequality? This question was answered by Hardy and Littlewood (see [19]) in 1934 for bilinear forms, and complemented by Praciano-Pereira (see [22]) in 1981 for m -linear forms and $p \geq 2m$ (and later by Dimant and Sevilla-Peris for $m < p < 2m$ (see [17])).

Theorem (Hardy–Littlewood/Praciano-Pereira [19, 22], 1934/1981). *Let $m \geq 2$ be a positive integer and $p \geq 2m$. For all m -linear forms $T : \ell_p^n \times \cdots \times \ell_p^n \rightarrow \mathbb{K}$ and all positive integers n ,*

$$(1.2) \quad \left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq (\sqrt{2})^{m-1} \|T\|.$$

Moreover, the exponent $2mp/(mp+p-2m)$ is optimal.

Theorem (Hardy–Littlewood/Dimant–Sevilla-Peris [19, 17], 1934/2014). *Let $m \geq 2$ be a positive integer and $m < p < 2m$. For all m -linear forms $T : \ell_p^n \times \cdots \times \ell_p^n \rightarrow \mathbb{K}$ and all positive integers n ,*

$$\left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^{\frac{p}{p-m}} \right)^{\frac{p-m}{p}} \leq (\sqrt{2})^{m-1} \|T\|.$$

Moreover, the exponent $p/(p-m)$ is optimal.

From now on, if f is a function, we define $f(\infty) := \lim_{p \rightarrow \infty} f(p)$ whenever it makes sense. In this fashion note that the Hardy–Littlewood/Praciano-Pereira inequality encompasses the Bohnenblust–Hille inequality.

To the best of our knowledge, the case $p \leq m$ was only explored for the case of Hilbert spaces ($p = 2$, see [10, Corollary 5.20] and [12]) and the case $p = \infty$ was explored in [11]. In [10, Corollary 5.20] it is shown that for $p = 2$ the inequality has an extra power of n in its right hand side. Other natural questions are how the the Hardy–Littlewood/Praciano-Pereira and Hardy–Littlewood/Dimant–Sevilla-Peris theorems behave if we replace the optimal exponents $2mp/(mp+p-2m)$ and $p/(p-m)$ by a smaller value r . More precisely, what power of n will appear, depending on r, m, p ?

Our main results answer this question (see Theorem 1.1) and extends [10, Corollary 5.20] to $1 \leq p \leq m$ (see Theorem 1.1(a) and Proposition 3.1).

The main result of this note is the following:

Theorem 1.1. *Let $m \geq 2$ be a positive integer.*

- (a) *If $(r, p) \in ([1, 2] \times [2, 2m]) \cup ([1, \infty) \times [2m, \infty])$, then there is a constant $D_{m,r,p}^{\mathbb{K}} > 0$ (not depending on n) such that*

$$\left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^r \right)^{\frac{1}{r}} \leq D_{m,r,p}^{\mathbb{K}} n^{\max\{\frac{2mr+2mp-mpr-pr}{2pr}, 0\}} \|T\|$$

for all m -linear forms $T : \ell_p^n \times \dots \times \ell_p^n \rightarrow \mathbb{K}$ and all positive integers n . Moreover, the exponent $\max\{(2mr + 2mp - mpr - pr)/2pr, 0\}$ is optimal.

- (b) *If $(r, p) \in [2, \infty) \times (m, 2m]$, then there is a constant $D_{m,r,p}^{\mathbb{K}} > 0$ (not depending on n) such that*

$$\left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^r \right)^{\frac{1}{r}} \leq D_{m,r,p}^{\mathbb{K}} n^{\max\{\frac{p+mr-rp}{pr}, 0\}} \|T\|$$

for all m -linear forms $T : \ell_p^n \times \dots \times \ell_p^n \rightarrow \mathbb{K}$ and all positive integers n . Moreover, the exponent $\max\{(p + mr - rp)/pr, 0\}$ is optimal.

Remark 1.2. *The first item of the above theorem recovers [10, Corollary 5.20(i)] (just make $p = 2$) and [11, Proposition 5.1].*

2. THE PROOF

2.1. First part: preparatory results. We begin by recalling a generalization of the Kahane–Salem–Zygmund inequality which is an extension of a result due to Boas ([7]) that will be useful in the proof of the optimality of the exponents:

Generalized Kahane–Salem–Zygmund inequality (see [2]). *Let $m, n \geq 1$, let $p \in [1, \infty]$, and let*

$$\alpha(p) = \begin{cases} \frac{1}{2} - \frac{1}{p} & \text{if } p \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

There is an universal constant C_m (depending only on m) and there exists an m -linear form $A : \ell_p^n \times \dots \times \ell_p^n \rightarrow \mathbb{K}$ of the form

$$(2.1) \quad A(z^{(1)}, \dots, z^{(m)}) = \sum_{i_1, \dots, i_m=1}^n \pm z_{i_1}^{(1)} \dots z_{i_m}^{(m)}$$

such that

$$\|A\| \leq C_m n^{\frac{1}{2} + m\alpha(p)}.$$

Henceforth, for all $p \in [1, \infty]$, we represent its conjugate number by p^* , i.e., $\frac{1}{p} + \frac{1}{p^*} = 1$. We will also use the following notation for the best known upper estimates of the

Bohnenblust–Hille inequality:

$$\begin{aligned}\eta_{\mathbb{C},m} &:= \prod_{j=2}^m \Gamma\left(2 - \frac{1}{j}\right)^{\frac{j}{2-2j}}, \\ \eta_{\mathbb{R},m} &:= \prod_{j=2}^m 2^{\frac{1}{2j-2}} \quad \text{for } m \leq 13, \\ \eta_{\mathbb{R},m} &:= 2^{\frac{446381}{55440} - \frac{m}{2}} \prod_{j=14}^m \left(\frac{\Gamma(\frac{3}{2} - \frac{1}{j})}{\sqrt{\pi}}\right)^{\frac{j}{2-2j}} \quad \text{for } m \geq 14.\end{aligned}$$

Let $1 \leq q \leq r \leq \infty$ and E be a Banach space. We recall that an m -linear form $S : E \times \cdots \times E \rightarrow \mathbb{K}$ is called multiple $(r; q)$ -summing if there is a constant $C > 0$ such that

$$\left\| (S(x_{j_1}^{(1)}, \dots, x_{j_m}^{(m)}))_{j_1, \dots, j_m=1}^n \right\|_{\ell_r} \leq C \sup_{\varphi \in B_{E^*}} \left(\sum_{j=1}^n |\varphi(x_j^{(1)})|^q \right)^{\frac{1}{q}} \cdots \sup_{\varphi \in B_{E^*}} \left(\sum_{j=1}^n |\varphi(x_j^{(m)})|^q \right)^{\frac{1}{q}}$$

for all positive integers n .

2.2. Second part: the proof. (a) Let us consider first $(r, p) \in [1, 2] \times [2, 2m)$. From now on $T : \ell_p^n \times \cdots \times \ell_p^n \rightarrow \mathbb{K}$ is an m -linear form. Since

$$\sup_{\varphi \in B_{(\ell_p^n)^*}} \sum_{j=1}^n |\varphi(e_j)| = n n^{-\frac{1}{p^*}} = n^{\frac{1}{p}}$$

and since T is multiple $(2m/(m+1); 1)$ -summing (recall that from the Bohnenblust–Hille inequality we know that all continuous m -linear forms are multiple $(2m/(m+1); 1)$ -summing with constant $\eta_{\mathbb{K},m}$), we conclude that

$$(2.2) \quad \left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq \eta_{\mathbb{K},m} \|T\| n^{\frac{m}{p}}.$$

Therefore, if $1 \leq r < 2m/(m+1)$, using the Hölder inequality and (2.2), we have

$$\begin{aligned}& \left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^r \right)^{\frac{1}{r}} \\ & \leq \left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \left(\sum_{j_1, \dots, j_m=1}^n |1|^{\frac{2mr}{2m-rm-r}} \right)^{\frac{2m-rm-r}{2mr}} \\ & = \left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} (n^m)^{\frac{2m-rm-r}{2mr}} \\ & \leq \eta_{\mathbb{K},m} \|T\| n^{\frac{m}{p}} n^{\frac{2m-rm-r}{2r}} \\ & = \eta_{\mathbb{K},m} n^{\frac{2mr+2mp-mpr-pr}{2pr}} \|T\|.\end{aligned}$$

Now we consider the case $2m/(m+1) \leq r \leq 2$. From the proof of [4, Theorem 3.2(i)] and from [5, Theorem 1.1] we know that, for all $2m/(m+1) \leq r \leq 2$ and all Banach spaces E , every continuous m -linear operator $S : E \times \cdots \times E \rightarrow \mathbb{K}$ is multiple

$(r; 2mr/(mr + 2m - r))$ -summing with constant $(\sigma_{\mathbb{K}})^{\frac{(m-1)(mr+r-2m)}{r}} (\eta_{\mathbb{K},m})^{\frac{2m-rm}{r}}$ where $\sigma_{\mathbb{R}} = \sqrt{2}$ and $\sigma_{\mathbb{K}} = 2/\sqrt{\pi}$. Therefore

$$(2.3) \quad \left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^r \right)^{\frac{1}{r}} \leq (\sigma_{\mathbb{K}})^{\frac{(m-1)(mr+r-2m)}{r}} (\eta_{\mathbb{K},m})^{\frac{2m-rm}{r}} \|T\| \left[\left(\sup_{\varphi \in B_{(\ell_p^n)^*}} \sum_{j=1}^n |\varphi(e_j)|^{\frac{2mr}{mr+2m-r}} \right)^{\frac{mr+2m-r}{2mr}} \right]^m.$$

Since $1 \leq 2mr/(mr + 2m - r) \leq 2m/(2m - 1) = (2m)^* < p^*$, we have

$$(2.4) \quad \left(\sup_{\varphi \in B_{(\ell_p^n)^*}} \sum_{j=1}^n |\varphi(e_j)|^{\frac{2mr}{mr-r+2m}} \right)^{\frac{mr-r+2m}{2mr}} = \left(n(n^{-\frac{1}{p^*}})^{\frac{2mr}{mr-r+2m}} \right)^{\frac{mr-r+2m}{2mr}} = n^{\frac{2mr+2mp-mpr-pr}{2mpr}}$$

and finally, from (2.3) and (2.4), we obtain

$$\left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^r \right)^{\frac{1}{r}} \leq (\sigma_{\mathbb{K}})^{\frac{(m-1)(mr+r-2m)}{r}} (\eta_{\mathbb{K},m})^{\frac{2m-rm}{r}} n^{\frac{2mr+2mp-mpr-pr}{2pr}} \|T\|.$$

Now we prove the optimality of the exponents. Suppose that the theorem is valid for an exponent s , i.e.,

$$\left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^r \right)^{\frac{1}{r}} \leq D_{m,r,p}^{\mathbb{K}} n^s \|T\|.$$

Since $p \geq 2$, from the Generalized Kahane–Salem–Zygmund inequality (using the m -linear form (2.1)) we have

$$n^{\frac{m}{r}} \leq C_m D_{m,r,p}^{\mathbb{K}} n^s n^{\frac{m+1}{2} - \frac{m}{p}}$$

and thus, making $n \rightarrow \infty$, we obtain

$$s \geq \frac{2mr+2mp-mpr-pr}{2pr}.$$

The case $(r, p) \in [1, 2mp/(mp + p - 2m)] \times [2m, \infty]$ is analogous. In fact, from the Hardy–Littlewood/Praciano-Pereira inequality and [5, Theorem 1.1] we know that

$$(2.5) \quad \left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq (\sigma_{\mathbb{K}})^{\frac{2m(m-1)}{p}} (\eta_{\mathbb{K},m})^{\frac{p-2m}{p}} \|T\|.$$

Therefore, from Hölder's inequality and (2.5), we have
(2.6)

$$\begin{aligned}
& \left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^r \right)^{\frac{1}{r}} \\
& \leq \left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \left(\sum_{j_1, \dots, j_m=1}^n |1|^{\frac{2mpr}{2mp+2mr-mpr-pr}} \right)^{\frac{2mp+2mr-mpr-pr}{2mpr}} \\
& = \left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} (n^m)^{\frac{2mp+2mr-mpr-pr}{2mpr}} \\
& \leq (\sigma_{\mathbb{K}})^{\frac{2m(m-1)}{p}} (\eta_{\mathbb{K}, m})^{\frac{p-2m}{p}} n^{\frac{2mp+2mr-mpr-pr}{2pr}} \|T\|.
\end{aligned}$$

Since $p \geq 2m$, the optimality of the exponent is obtained *ipsis litteris* as in the previous case.

If $(r, p) \in (2mp/(mp + p - 2m), \infty) \times [2m, \infty]$ we have

$$\frac{2mr+2mp-mpr-pr}{2pr} < 0$$

and

$$\begin{aligned}
\left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^r \right)^{\frac{1}{r}} & \leq \left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \\
& \leq D_{m, \frac{2mp}{mp+p-2m}, p}^{\mathbb{K}} \|T\| \\
& = D_{m, \frac{2mp}{mp+p-2m}, p}^{\mathbb{K}} \|T\| n^{\max\{\frac{2mr+2mp-mpr-pr}{2pr}, 0\}}.
\end{aligned}$$

In this case the optimality of the exponent $\max\{(2mr + 2mp - mpr - pr)/2pr, 0\}$ is immediate, since one can easily verify that no negative exponent of n is possible.

(b) Let us first consider $(r, p) \in [2, p/(p - m)] \times (m, 2m]$. Define

$$q = \frac{mr}{r-1}$$

and note that $q \leq 2m$ and $r = q/(q - m)$. Since $q/(q - m) = r \leq p/(p - m)$ we have $p \leq q$. Then $m < p \leq q \leq 2m$. Note that

$$q^* = \frac{mr}{mr+1-r}.$$

Since $m < q \leq 2m$, by the Hardy-Littlewood/Dimant-Sevilla-Peris inequality and using [17, Section 5] we know that every continuous m -linear operator on any Banach space E is multiple $(q/(q - m); q^*)$ -summing with constant $(\sqrt{2})^{m-1}$, i.e., multiple $(r; mr/(mr + 1 - r))$ -summing with constant $(\sqrt{2})^{m-1}$. So for $T : \ell_p^n \times \dots \times \ell_p^n \rightarrow \mathbb{K}$

we have (since $q^* \leq p^*$),

$$\begin{aligned}
& \left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^r \right)^{\frac{1}{r}} \\
& \leq (\sqrt{2})^{m-1} \|T\| \left[\left(\sup_{\varphi \in B_{(\ell_p^n)^*}} \sum_{j=1}^n |\varphi(e_j)|^{\frac{mr}{mr+1-r}} \right)^{\frac{mr+1-r}{mr}} \right]^m \\
& = (\sqrt{2})^{m-1} \|T\| \left[\left(n(n^{-\frac{1}{p^*}})^{\frac{mr}{mr+1-r}} \right)^{\frac{mr+1-r}{mr}} \right]^m \\
& = (\sqrt{2})^{m-1} \|T\| n^{\frac{p+mr-rp}{pr}}.
\end{aligned}$$

Note that if we have tried to use above an argument similar to (2.6), via Hölder's inequality, we would obtain worse exponents. Now we prove the optimality following the lines of [17]. Defining $R : \ell_p^n \times \dots \times \ell_p^n \rightarrow \mathbb{K}$ by $R(x^{(1)}, \dots, x^{(m)}) = \sum_{j=1}^n x_j^{(1)} \dots x_j^{(m)}$, from Hölder's inequality we can easily verify that

$$\|R\| \leq n^{1-\frac{m}{p}}.$$

So if the theorem holds for n^s , plugging the m -linear form R into the inequality we have

$$n^{\frac{1}{r}} \leq D_{m,r,p}^{\mathbb{K}} n^s n^{1-\frac{m}{p}}$$

and thus, by making $n \rightarrow \infty$, we obtain

$$s \geq \frac{p+mr-rp}{pr}.$$

If $(r, p) \in (p/(p-m), \infty) \times (m, 2m]$ we have

$$\frac{p+mr-rp}{pr} < 0$$

and

$$\begin{aligned}
\left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^r \right)^{\frac{1}{r}} & \leq \left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^{\frac{p}{p-m}} \right)^{\frac{p-m}{p}} \\
& \leq (\sqrt{2})^{m-1} \|T\| \\
& = (\sqrt{2})^{m-1} \|T\| n^{\max\{\frac{p+mr-rp}{pr}, 0\}}
\end{aligned}$$

In this case the optimality of the exponent $\max\{(p+mr-rp)/pr, 0\}$ is immediate, since one can easily verify that no negative exponent of n is possible.

Remark 2.1. Observing the proof of Theorem 1.1 we conclude that the optimal constant $D_{m,r,p}^{\mathbb{K}}$ satisfies:

$$D_{m,r,p}^{\mathbb{K}} \leq \begin{cases} \eta_{\mathbb{K},m} & \text{if } (r, p) \in [1, \frac{2m}{m+1}] \times [2, 2m), \\ (\sigma_{\mathbb{K}})^{\frac{(m-1)(mr+r-2m)}{r}} (\eta_{\mathbb{K},m})^{\frac{2m-rm}{r}} & \text{if } (r, p) \in (\frac{2m}{m+1}, 2] \times [2, 2m), \\ (\sigma_{\mathbb{K}})^{\frac{2m(m-1)}{p}} (\eta_{\mathbb{K},m})^{\frac{p-2m}{p}} & \text{if } (r, p) \in [1, \infty) \times [2m, \infty), \\ (\sqrt{2})^{m-1} & \text{if } (r, p) \in [2, \infty) \times (m, 2m]. \end{cases}$$

If $p > 2m^3 - 4m^2 + 2m$, using [3] we can improve $(\sigma_{\mathbb{K}})^{\frac{2m(m-1)}{p}} (\eta_{\mathbb{K},m})^{\frac{p-2m}{p}}$ to $\eta_{\mathbb{K},m}$.

3. FINAL COMMENTS AND RESULTS

In this section we obtain partial answers for the cases not covered by our main theorem, i.e., the cases $(r, p) \in [1, 2] \times [1, 2)$ and $(r, p) \in (2, \infty) \times [1, m]$.

Proposition 3.1. *Let $m \geq 2$ be a positive integer.*

(a) *If $(r, p) \in [1, 2] \times [1, 2)$, then there is a constant $D_{m,r,p}^{\mathbb{K}} > 0$ such that*

$$(3.1) \quad \left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^r \right)^{\frac{1}{r}} \leq D_{m,r,p}^{\mathbb{K}} n^{\frac{2mr+2mp-mpr-pr}{2pr}} \|T\|$$

for all m -linear forms $T : \ell_p^n \times \dots \times \ell_p^n \rightarrow \mathbb{K}$ and all positive integers n . Moreover the optimal exponent of n is not smaller than $(2m-r)/2r$.

(b) *If $(r, p) \in (2, \infty) \times [1, m]$, then there is a constant $D_{m,r,p}^{\mathbb{K}} > 0$ such that*

$$\left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^r \right)^{\frac{1}{r}} \leq \begin{cases} D_{m,r,p}^{\mathbb{K}} n^{\frac{2m-p+\varepsilon}{pr}} \|T\| & \text{if } p > 2 \\ D_{m,r,p}^{\mathbb{K}} n^{\frac{2m-p}{pr}} \|T\| & \text{if } p = 2 \end{cases}$$

for all m -linear forms $T : \ell_p^n \times \dots \times \ell_p^n \rightarrow \mathbb{K}$ and all positive integers n and all $\varepsilon > 0$. Moreover the optimal exponent of n is not smaller than $(2mr+2mp-mpr-pr)/2pr$ if $2 \leq p \leq m$. In the case $1 \leq p \leq 2$ the optimal exponent of n is not smaller than $(2m-r)/2r$.

Proof. (a) The proof of (3.1) is the same of the proof of Theorem 1.1(a). The estimate for the bound of the optimal exponent also uses the Generalized Kahane–Salem–Zygmund inequality. Since $p \leq 2$ we have

$$n^{\frac{m}{r}} \leq C_m D_{m,r,p}^{\mathbb{K}} n^s n^{\frac{1}{2}}$$

and thus, by making $n \rightarrow \infty$,

$$s \geq \frac{2m-r}{2r}.$$

(b) Let $\delta = 0$ if $p = 2$ and $\delta > 0$ if $p > 2$. First note that every continuous m -linear form on ℓ_p spaces is obviously multiple $(\infty; p^* - \delta)$ -summing and also multiple $(2; 2m/(2m-1))$ -summing (this is a consequence of the Hardy–Littlewood inequality and [17, Section 5]). Using [10, Proposition 4.3] we conclude that every continuous m -linear form on ℓ_p spaces is multiple $(r; mpr/(2m+mpr-mr-p+\varepsilon))$ -summing for all $\varepsilon > 0$ (and $\varepsilon = 0$ if $p = 2$). Therefore, there exist $D_{m,r,p}^{\mathbb{K}} > 0$ such that

$$\begin{aligned} & \left(\sum_{j_1, \dots, j_m=1}^n |T(e_{j_1}, \dots, e_{j_m})|^r \right)^{\frac{1}{r}} \\ & \leq D_{m,r,p}^{\mathbb{K}} \left[\left(n(n^{-\frac{1}{p^*}})^{\frac{mpr}{2m+mpr-mr-p+\varepsilon}} \right)^{\frac{2m+mpr-mr-p+\varepsilon}{mpr}} \right]^m \|T\| \\ & = D_{m,r,p}^{\mathbb{K}} n^{\frac{2m+mpr-mr-p+\varepsilon}{pr}} (n^{\frac{1}{p}-1})^m \|T\| \\ & = D_{m,r,p}^{\mathbb{K}} n^{\frac{2m-p+\varepsilon}{pr}} \|T\|. \end{aligned}$$

The bounds for the optimal exponents are obtained via the Generalized Kahane–Salem–Zygmund inequality as in the previous cases. \square

Remark 3.2. *Item (b) of the Proposition 3.1 with $p = 2$ recovers [10, Corollary 5.20(ii)].*

We believe that the remaining cases (those in which we do not have achieved the optimality of the exponents) are interesting for further investigation trying to have a full panorama, covering all cases with optimal estimates.

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